J. Fluid Mech. (1971), *vol.* 47, *part* 3, *pp.* 537–546
Printed in Great Britain

The effect of oscillation on flat plate heat transfer

By HIROSHI ISHIGAKI
National Aerospace Laboratory, Kakuda Branch, Miyagi, Japan

(Received 16 September 1970)

The time-mean heat transfer of the incompressible laminar boundary layer on a flat plate under the influence of oscillation is studied analytically. Flow oscillation amplitude outside the boundary layer is assumed constant along the surface and the viscous dissipation effect is considered. First, the small velocity–amplitude case is treated and the approximate formulae are obtained in the extreme cases when the frequency is low and high. Next, the finite velocity–amplitude case is treated under the condition of high frequency and it is found that the formulae obtained for the small amplitude and high frequency case are also valid. These results show that, when the oscillation is of high frequency, the time-mean heat flux to the wall can be several times as large as that without oscillation. This is due wholly to the viscous dissipation effect combined with oscillation.

1. Introduction

In a recent paper Ishigaki (1971) studied the effect of oscillation on the time-mean skin friction and adiabatic wall temperature of a flat plate, and confirmed the earlier result by Stuart (1955) that viscous dissipation has a large effect on the time-mean temperature field when the oscillation is of high frequency. In the present paper the corresponding heat transfer problem is studied and it is shown how the viscous dissipation combined with oscillation affects the time-mean heat transfer from or to the wall.

Heat transfer under the influence of vibration and flow oscillation (also sound) has been the subject of much research since the early 1950's. One of the practical problems which inspired interest in the effect of oscillation on heat transfer is encountered in liquid rocket and turbo-jet engines. When a high frequency combustion oscillation occurs in such engines, the heat flux to the engine wall increases abruptly and the wall temperature often rises to the melting point of material. This causes the failure of liquid rocket engine combustion chambers, propellant injectors and turbo-jet engine after-burners shortly after the onset of, so-called, screaming or screeching combustion.

Theoretical study on a fluctuating heat transfer of a periodic boundary layer with an on-coming stream has been made by many authors, e.g. Lighthill (1954), Ostrach (1955), Illingworth (1958). As to the time-mean heat transfer, studies were made for flat plate flow with small amplitude and low frequency oscillation by Moore & Ostrach (1957), Kestin, Maeder & Wang (1961). The time-mean heat transfer in a wedge-type flow with small amplitude oscillation was studied by Gersten (1965). The effect of a longitudinal acoustic field on the time-mean heat transfer in a parallel plate channel was studied by Keith & Purdy (1967). The time-mean mass transfer problem in fully developed flow in a tube with a small periodic pressure gradient was studied by Fagela-Alabastro & Hellums (1969).

In these time-mean studies except that of Moore & Ostrach, the viscous dissipation effect was neglected. Therefore the effect of flow oscillation on the time-mean temperature field of incompressible fluid brings out the following two additional heat flux: (i) due to the convection by the secondary flow induced by the Reynolds stresses (e.g. $\overline{u'v'}$); (ii) due to the correlations of fluctuating velocities and temperature, e.g. $\overline{u'T'}$; here u', v' are the fluctuating velocities, T' is the fluctuating temperature and the over-bar denotes an average with respect to time. Inferring from the above-mentioned works it may be said that the effect of these two factors on the time-mean temperature field is rather small. In this paper the third factor of viscous dissipation is considered simultaneously and its predominance at high frequency is shown. In the paper by Moore & Ostrach the treatment was restricted to low frequency oscillation, so that the effect of viscous dissipation is of comparable order of magnitude with the above two factors.

2. Basic equations

Let us consider a two-dimensional unsteady laminar boundary layer with viscous dissipation of an incompressible fluid with constant properties. Let x and y denote the co-ordinates parallel and normal to the wall, and u, v the corresponding velocity components. In addition, let T denote temperature, t time, ν kinematic viscosity, κ thermal diffusivity, c specific heat, and $U(x,t)$ the external flow velocity.

We define functions ψ and θ by

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x, \quad \theta = (T-T_\infty)/(T_w-T_\infty), \tag{1}$$

in which T_w denotes the wall temperature and T_∞ the external flow temperature (both are constant). Then the boundary-layer equations for velocity and temperature may be written in the forms

$$\frac{\partial^2\psi}{\partial t\,\partial y}+\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^3\psi}{\partial y^3}, \tag{2}$$

$$\frac{\partial\theta}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} = \kappa\frac{\partial^2\theta}{\partial y^2}+\frac{\nu}{c(T_w-T_\infty)}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2, \tag{3}$$

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0, \quad \theta = 1 \quad \text{at} \quad y=0, \quad \frac{\partial\psi}{\partial y} = U(x,t), \quad \theta = 0 \quad \text{as} \quad y\to\infty.$$

Letting ω denote frequency and ϵ velocity–amplitude ratio, we shall confine our attention to the function $U(x,t)$ of the form

$$U = U_\infty(1+\epsilon e^{i\omega t}), \tag{4}$$

which is independent of x and in which only the real part has the physical meaning. Further restriction on either ϵ or ω may be needed when these equations are tackled.

3. Small velocity-amplitude case

When ϵ is smaller than unity, we may develop the function ψ and θ in the forms

$$\left.\begin{aligned} \psi(x,y,t) &= \psi_0(x,y)+\epsilon\psi_1(x,y)\,e^{i\omega t}+\epsilon^2\{\psi_s(x,y)+\psi_2(x,y)\,e^{2i\omega t}\}+O(\epsilon^3),\\ \theta(x,y,t) &= \theta_0(x,y)+\epsilon\theta_1(x,y)\,e^{i\omega t}+\epsilon^2\{\theta_s(x,y)+\theta_2(x,y)\,e^{2i\omega t}\}+O(\epsilon^3), \end{aligned}\right\} \quad (5)$$

where only the real parts are to be taken. Substituting (4), (5) into (2), (3) and equating the same order of ϵ, sets of equations are obtained. The equations for ψ_0 and θ_0 are the steady-state equations and the solutions are the following well-known functions:

$$\psi_0 = (2\nu U_\infty x)^{\frac{1}{2}}f(\eta), \quad \theta_0 = h(\eta)+\Gamma s(\eta), \quad \eta = (U_\infty/2\nu x)^{\frac{1}{2}}y, \quad (6)$$

in which $\Gamma = U_\infty^2/\{2c(T_w-T_\infty)\}$. The equation for θ_1 is

$$\frac{\partial\theta_1}{\partial t}+\frac{\partial\psi_0}{\partial y}\frac{\partial\theta_1}{\partial x}+\frac{\partial\psi_1}{\partial y}\frac{\partial\theta_0}{\partial x}-\frac{\partial\psi_0}{\partial x}\frac{\partial\theta_1}{\partial y}-\frac{\partial\psi_1}{\partial x}\frac{\partial\theta_0}{\partial y} = \kappa\frac{\partial^2\theta_1}{\partial y^2}+\frac{2\nu}{c(T_w-T_\infty)}\frac{\partial^2\psi_0}{\partial y^2}\frac{\partial^2\psi_1}{\partial y^2}, \quad (7)$$

$$\theta_1 = 0 \quad \text{at} \quad y=0, \quad \theta_1 = 0 \quad \text{as} \quad y\to\infty.$$

Approximate solutions have been obtained in the extreme cases when the frequency parameter $\sigma = \omega x/U_\infty$ is small and large. For small value of σ the results of Moore (1951) and Ostrach (1955) can be particularized as

$$\psi_1 = (2\nu U_\infty x)^{\frac{1}{2}}\sum_{n=0}(i\sigma)^n g_{l,n}(\eta), \; \theta_1 = \sum_{n=0}(i\sigma)^n\{k_{l,n}(\eta)+\Gamma w_{l,n}(\eta)\}. \quad (8)$$

For large value of σ, the method due to Illingworth (1958) may be appropriate. If $\alpha = (2i\sigma)^{-\frac{1}{2}}$ and $\beta = (i\omega/\nu)^{\frac{1}{2}}y$, ψ_1 may be written

$$\psi_1 = U_\infty(\nu/i\omega)^{\frac{1}{2}}\sum_{n=0}\alpha^n g_{h,n}(\beta), \quad \theta_1 = \sum_{n=0}\alpha^n\{k_{h,n}(\beta)+\Gamma w_{h,n}(\beta)\}. \quad (9)$$

Provided that α is small and β is not too large, the following approximations are made when $\eta = \alpha\beta$ is considered:

$$\left.\begin{aligned} f(\alpha\beta) &= \tfrac{1}{2}\alpha^2\beta^2 f''(0)+O(\alpha^5), \quad h(\alpha\beta) = 1+\alpha\beta h'(0)+O(\alpha^4),\\ s(\alpha\beta) &= \alpha\beta s'(0)+\tfrac{1}{2}\alpha^2\beta^2 s''(0)+O(\alpha^4) \end{aligned}\right\} \quad (10)$$

in which primes denote differentiation. Substituting (9), (10) into (7), ordinary differential equations are obtained for each order of α, and the solutions which satisfy the boundary conditions at $\beta = 0$ are

$$k_{h,0} = k_{h,1} = k_{h,2} = 0, \quad k_{h,3} = \left\{\beta+\frac{Pr}{1-Pr}\left(\frac{2}{1-Pr}+\beta\right)e^{-\beta}-\frac{2Pr}{(1-Pr)^2}\,e^{-(Pr)^{\frac{1}{2}}\beta}\right\}h'(0),$$

$$w_{h,0} = 0, \quad w_{h,1} = \frac{4Pr}{1-Pr}\,(e^{-(Pr)^{\frac{1}{2}}\beta}-e^{-\beta})f''(0), \quad w_{h,2} = 0, \quad (11)$$

in which $Pr = \nu/\kappa$ is the Prandtl number. For the amplitude and phase angle of fluctuating heat transfer of order ϵ under the condition of negligible viscous dissipation, the reader may refer to the results of Illingworth and Gersten.

The equation for the time independent function θ_s is

$$\frac{\partial\psi_0}{\partial y}\frac{\partial\theta_s}{\partial x}+\frac{\partial\psi_s}{\partial y}\frac{\partial\theta_0}{\partial x}-\frac{\partial\psi_0}{\partial x}\frac{\partial\theta_s}{\partial y}-\frac{\partial\psi_s}{\partial x}\frac{\partial\theta_0}{\partial y}+\frac{1}{2}\left\{\frac{\partial\psi_{1r}}{\partial y}\frac{\partial\theta_{1r}}{\partial x}+\frac{\partial\psi_{1i}}{\partial y}\frac{\partial\theta_{1i}}{\partial x}-\frac{\partial\psi_{1r}}{\partial x}\frac{\partial\theta_{1r}}{\partial y}-\frac{\partial\psi_{1i}}{\partial x}\frac{\partial\theta_{1i}}{\partial y}\right\}$$

$$=\kappa\frac{\partial^2\theta_s}{\partial y^2}+\frac{\nu}{c(T_w-T_\infty)}\left[2\frac{\partial^2\psi_0}{\partial y^2}\frac{\partial^2\psi_s}{\partial y^2}+\frac{1}{2}\left\{\left(\frac{\partial^2\psi_{1r}}{\partial y^2}\right)^2+\left(\frac{\partial^2\psi_{1i}}{\partial y^2}\right)^2\right\}\right], \tag{12}$$

$$\theta_s=0 \quad \text{at} \quad y=0, \quad \theta_s=0 \quad \text{as} \quad y\to\infty$$

where subscripts r and i denote the real and imaginary parts of a function. For small value of σ appropriate forms may be written as

$$\psi_s=(2\nu U_\infty x)^{\frac{1}{2}}\sum_{n=0}\sigma^{2n}G_{l,2n}(\eta), \quad \theta_s=\sum_{n=0}\sigma^{2n}\{K_{l,2n}(\eta)+\Gamma W_{l,2n}(\eta)\} \tag{13}$$

in which $K_{l,0}$ and $W_{l,0}$ are the following quasi-steady terms:

$$K_{l,0}=\tfrac{1}{16}(-\eta h'+\eta^2h''), \quad W_{l,0}=\tfrac{1}{16}(8s+7\eta s'+\eta^2 s'').$$

The equation for $K_{l,2}$ was solved by Gersten. The equation for $W_{l,2}$ is

$$\frac{1}{Pr}W''_{l,2}+fW'_{l,2}-4f'W_{l,2}=-5s'G_{l,2}-4f''G''_{l,2}+2g''_{l,0}g''_{l,2}-(g''_{l,1})^2+\tfrac{5}{2}g_{l,2}w'_{l,0}$$
$$+g'_{l,1}w_{l,1}-\tfrac{3}{2}g_{l,1}w'_{l,1}-2g'_{l,0}w_{l,2}+\tfrac{1}{2}g_{l,0}w'_{l,2}$$

and is solved numerically. For large value of σ we let

$$\theta_s(x,y)=K_h(x,y)+\Gamma W_h(x,y). \tag{14}$$

Substitutions of (9), (10) and (11) into (12) yield the equation for K_h as

$$\kappa\frac{\partial^2K_h}{\partial y^2}-\frac{\partial\psi_0}{\partial y}\frac{\partial K_h}{\partial x}+\frac{\partial\psi_0}{\partial x}\frac{\partial K_h}{\partial y}=\frac{3PrU_\infty h'(0)}{8(1-Pr)^2x\sigma^{\frac{3}{2}}}\left[\cos z\cdot e^{-z}+\left(1+\frac{1-Pr}{Pr}z\right)\sin z\cdot e^{-z}-e^{-2z}\right.$$
$$\left.-\{\cos Pr^{\frac{1}{2}}z+\sin Pr^{\frac{1}{2}}z\}e^{-Pr^{\frac{1}{2}}z}+\{\cos(1-Pr^{\frac{1}{2}})z-\sin(1-Pr^{\frac{1}{2}})z\}e^{-(1+Pr^{\frac{1}{2}})z}\right]+O(\sigma^{-3}), \tag{15}$$

$$K_h=0 \quad \text{at} \quad y=0, \quad K_h=0 \quad \text{as} \quad y\to\infty,$$

in which $z=(\omega/2\nu)^{\frac{1}{2}}y$. Then the particular solution K_{hp} is

$$K_{hp}=\frac{3Prh'(0)}{8(1-Pr)^2\sigma^{\frac{5}{2}}}\left[\{1+(1-Pr)z\}\cos z\cdot e^{-z}-\sin z\cdot e^{-z}-\tfrac{1}{2}Pre^{-2z}\right.$$
$$-\{\cos Pr^{\frac{1}{2}}z-\sin Pr^{\frac{1}{2}}z\}e^{-Pr^{\frac{1}{2}}z}+\frac{Pr}{(1+Pr)^2}\{(Pr+2Pr^{\frac{1}{2}}-1)\cos(1-Pr^{\frac{1}{2}})z$$
$$\left.+(Pr-2Pr^{\frac{1}{2}}-1)\sin(1-Pr^{\frac{1}{2}})z\}e^{-(1+Pr^{\frac{1}{2}})z}\right]+O(\sigma^{-4}). \tag{16}$$

Therefore the contribution of homogeneous solution to the time-mean heat transfer is of the order of $\sigma^{-\frac{5}{2}}$ and is neglected in the after heat transfer estimation. The equation for W_h is

$$\kappa\frac{\partial^2W_h}{\partial y^2}-\frac{\partial\psi_0}{\partial y}\frac{\partial W_h}{\partial x}+\frac{\partial\psi_0}{\partial x}\frac{\partial W_h}{\partial y}=-\frac{\sigma U_\infty}{x}e^{-2z}+O(\sigma^{-\frac{1}{2}}), \tag{17}$$

$$W_h=0 \quad \text{at} \quad y=0, \quad W_h=0 \quad \text{as} \quad y\to\infty.$$

The particular solution W_{hp} is

$$W_{hp}=-\tfrac{1}{2}Pr\,e^{-2z}+O(\sigma^{-\frac{3}{2}}). \tag{18}$$

Then the homogeneous solution W_{hq} may be expressed as

$$W_{hq} = Q(\eta)+O(\sigma^{-\frac{3}{2}}). \tag{19a}$$

The equation for $Q(\eta)$ is

$$(1/Pr)Q''+fQ' = 0,$$

$$Q = \tfrac{1}{2}Pr \text{ at } y = 0, \quad Q = 0 \text{ as } y\to\infty$$

and the solution is

$$Q = \tfrac{1}{2}Pr\,h(\eta), \tag{19b}$$

in which $h(\eta)$ appears in (6).

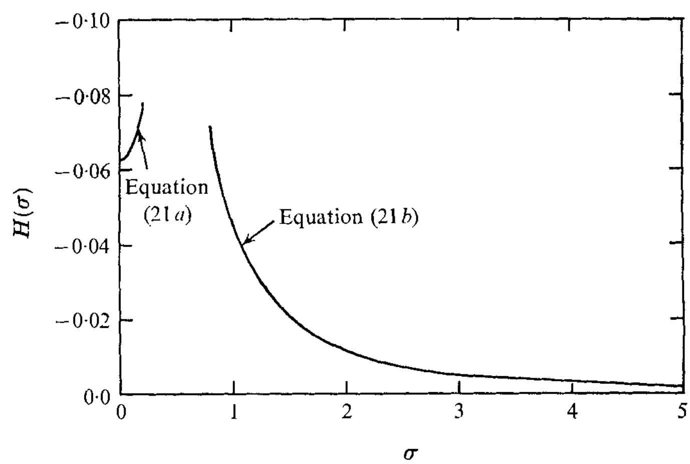

FIGURE 1. Plot of H with frequency parameter σ.

We can now obtain the approximate expression of the time-mean heat transfer. We write the time-mean heat flux from the wall as

$$\begin{aligned}\bar{q} &= -\frac{1}{2\pi}\int_0^{2\pi}\lambda(\partial T/\partial y)_{y=0}dt \\ &= -\lambda(T_w-T_\infty)(U_\infty/2\nu x)^{\frac{1}{2}}[h'(0)\{1+\epsilon^2H(\sigma)\}+\Gamma s'(0)\{1+\epsilon^2S(\sigma)\}]+O(\epsilon^4)\end{aligned} \tag{20}$$

in which λ is the thermal conductivity. For $Pr = 0\cdot72$ we have

$$H(\sigma) = -\tfrac{1}{16}-0\cdot4137\,\sigma^2+O(\sigma^4) \quad \text{(small } \sigma) \tag{21a}$$

$$= -\frac{0\cdot0459}{\sigma^2}+O(\sigma^{-\frac{5}{2}}) \quad \text{(large } \sigma) \tag{21b}$$

and

$$S(\sigma) = \tfrac{15}{16}+1\cdot3665\,\sigma^2+O(\sigma^4) \quad \text{(small } \sigma) \tag{22a}$$

$$= 2\cdot0135\,(\sigma)^{\frac{1}{2}}-0\cdot4247+O(\sigma^{-1}) \quad \text{(large } \sigma). \tag{22b}$$

Functions $H(\sigma)$ and $S(\sigma)$ are shown in figures 1 and 2, and in figure 2 the asymptotic value for very large σ, first term only in (22b), is also shown by a broken line. It can be seen that the contribution of $H(\sigma)$, which decreases abruptly with the increase of σ, to the time-mean heat flux is much smaller than that of $S(\sigma)$ for high frequency oscillations.

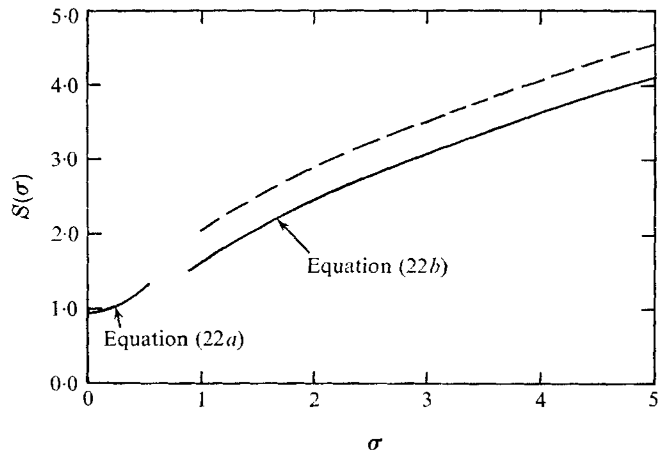

FIGURE 2. Plot of S with frequency parameter σ.

4. Finite velocity-amplitude case

Making some assumptions mainly on the frequency, another approach due to Lin (1957) is possible. In (3) we separate θ into two functions

$$\theta(x,y,t) = \theta_h(x,y,t) + \theta_p(x,y,t), \tag{23}$$

in which θ_h is a solution of the homogeneous equation and θ_p is a particular solution of (3). Moreover, we express the functions as the sum of a time-mean and a time-dependent component as

$$\left.\begin{aligned} \psi(x,y,t) &= \overline{\psi}(x,y) + \psi_t(x,y,t), \\ \theta_h(x,y,t) &= \overline{\theta}_h(x,y) + \theta_{ht}(x,y,t), \\ \theta_p(x,y,t) &= \overline{\theta}_p(x,y) + \theta_{pt}(x,y,t), \\ \overline{\psi}_t &= \overline{\theta}_{ht} = \overline{\theta}_{pt} = 0, \end{aligned}\right\} \tag{24}$$

where a bar over the symbols denotes time-mean quantities. Substituting (24) into (3) and taking its time average, the time-mean equations are obtained. Subtracting these from the full equation (3), the time-dependent equations are obtained. The equations and boundary conditions for $\overline{\theta}_h$ and θ_{ht} are

$$\frac{\partial\overline{\psi}}{\partial y}\frac{\partial\overline{\theta}_h}{\partial x} + \overline{\frac{\partial\psi_t}{\partial y}\frac{\partial\theta_{ht}}{\partial x}} - \frac{\partial\overline{\psi}}{\partial x}\frac{\partial\overline{\theta}_h}{\partial y} - \overline{\frac{\partial\psi_t}{\partial x}\frac{\partial\theta_{ht}}{\partial y}} = \kappa\frac{\partial^2\overline{\theta}_h}{\partial y^2}, \tag{25}$$

$$\begin{aligned} &\frac{\partial\theta_{ht}}{\partial t} + \frac{\partial\psi_t}{\partial y}\frac{\partial\overline{\theta}_h}{\partial x} + \frac{\partial\overline{\psi}}{\partial y}\frac{\partial\theta_{ht}}{\partial x} + \frac{\partial\psi_t}{\partial y}\frac{\partial\theta_{ht}}{\partial x} - \overline{\frac{\partial\psi_t}{\partial y}\frac{\partial\theta_{ht}}{\partial x}} \\ &\qquad - \left\{\frac{\partial\psi_t}{\partial x}\frac{\partial\overline{\theta}_h}{\partial y} + \frac{\partial\overline{\psi}}{\partial x}\frac{\partial\theta_{ht}}{\partial y} + \frac{\partial\psi_t}{\partial x}\frac{\partial\theta_{ht}}{\partial y} - \overline{\frac{\partial\psi_t}{\partial x}\frac{\partial\theta_{ht}}{\partial y}}\right\} = \kappa\frac{\partial^2\theta_{ht}}{\partial y^2}, \end{aligned} \tag{26}$$

$$\overline{\theta}_h = 1, \quad \theta_{ht} = 0 \quad \text{at} \quad y = 0, \quad \overline{\theta}_h = \theta_{ht} = 0 \quad \text{as} \quad y \to \infty.$$

The equations and boundary conditions for $\overline{\theta}_p$ and θ_{pt} are

$$\frac{\partial\overline{\psi}}{\partial y}\frac{\partial\overline{\theta}_p}{\partial x} + \overline{\frac{\partial\psi_t}{\partial y}\frac{\partial\theta_{pt}}{\partial x}} - \frac{\partial\overline{\psi}}{\partial x}\frac{\partial\overline{\theta}_p}{\partial y} - \overline{\frac{\partial\psi_t}{\partial x}\frac{\partial\theta_{pt}}{\partial y}} = \kappa\frac{\partial^2\overline{\theta}_p}{\partial y^2} + \frac{\nu}{c(T_w - T_\infty)}\left\{\left(\frac{\partial^2\overline{\psi}}{\partial y^2}\right)^2 + \overline{\left(\frac{\partial^2\psi_t}{\partial y^2}\right)^2}\right\}, \tag{27}$$

$$\frac{\partial\theta_{pt}}{\partial t}+\frac{\partial\psi_t}{\partial y}\frac{\partial\bar{\theta}_p}{\partial x}+\frac{\partial\bar{\psi}}{\partial y}\frac{\partial\theta_{pt}}{\partial x}+\frac{\partial\psi_t}{\partial y}\frac{\partial\theta_{pt}}{\partial x}-\overline{\frac{\partial\psi_t}{\partial y}\frac{\partial\theta_{pt}}{\partial x}}-\left\{\frac{\partial\psi_t}{\partial x}\frac{\partial\bar{\theta}_p}{\partial y}+\frac{\partial\bar{\psi}}{\partial x}\frac{\partial\theta_{pt}}{\partial y}\right.$$
$$\left.+\frac{\partial\psi_t}{\partial x}\frac{\partial\theta_{pt}}{\partial y}-\overline{\frac{\partial\psi_t}{\partial x}\frac{\partial\theta_{pt}}{\partial y}}\right\}=\kappa\frac{\partial^2\theta_{pt}}{\partial y^2}+\frac{\nu}{c(T_w-T_\infty)}\left\{2\frac{\partial^2\bar{\psi}}{\partial y^2}\frac{\partial^2\psi_t}{\partial y^2}+\left(\frac{\partial^2\psi_t}{\partial y^2}\right)^2-\overline{\left(\frac{\partial^2\psi_t}{\partial y^2}\right)^2}\right\}, \tag{28}$$

$$\bar{\theta}_p=\theta_{pt}=0 \quad \text{at} \quad y=0, \quad \bar{\theta}_p=\theta_{pt}=0 \quad \text{as} \quad y\to\infty.$$

In the case of high frequency, under the assumptions

$$\sigma \gg 1, \epsilon \tag{29}$$

the approximate expressions of $\bar{\psi}$ and ψ_t are given as

$$\left.\begin{aligned}\bar{\psi}&=\psi_0=(2\nu U_\infty x)^{\frac{1}{2}}f(\eta),\\ \psi_t&=\epsilon U_\infty(\nu/i\omega)^{\frac{1}{2}}\{y(i\omega/\nu)^{\frac{1}{2}}-1+\exp(-y(i\omega/\nu)^{\frac{1}{2}})\}e^{i\omega t}.\end{aligned}\right\} \tag{30}$$

In (26) the relative order of magnitude between θ_{ht} and $\bar{\theta}_h$ can be estimated as

$$\frac{\partial\theta_{ht}}{\partial t}\sim\frac{\partial\psi_t}{\partial y}\frac{\partial\bar{\theta}_h}{\partial x} \quad \text{or} \quad \frac{\theta_{ht}}{\bar{\theta}_h}\sim\frac{\epsilon}{\sigma}\ll 1.$$

Moreover, in (25), assuming

$$\frac{\partial\bar{\psi}}{\partial y}\frac{\partial\bar{\theta}_h}{\partial x}\gg\overline{\frac{\partial\psi_t}{\partial y}\frac{\partial\theta_{ht}}{\partial x}} \quad \text{or} \quad \sigma\gg\epsilon^2, \tag{31}$$

the approximate solution is obtained as

$$\bar{\theta}_h=h(\eta). \tag{32}$$

There it follows that the effect of high frequency oscillation on $\bar{\theta}_h$ may be negligible to the first approximation. Then, after neglecting the convective terms in (26) under the assumptions (29), the equation for θ_{ht} becomes

$$\kappa\frac{\partial^2\theta_{ht}}{\partial y^2}-\frac{\partial\theta_{ht}}{\partial t}=\frac{\partial\psi_t}{\partial y}\frac{\partial\bar{\theta}_h}{\partial x}. \tag{33}$$

An approximate method similar to (9), (10) gives the same result as the small amplitude case, that

$$\theta_{ht}=\frac{\epsilon h'(0)}{(2i\sigma)^{\frac{3}{2}}}\left[y(i\omega/\nu)^{\frac{1}{2}}+\frac{Pr}{(1-Pr)^2}\{2+(1-Pr)y(i\omega/\nu)^{\frac{1}{2}}\}\exp(-y(i\omega/\nu)^{\frac{1}{2}})\right.$$
$$\left.-\frac{2Pr}{(1-Pr)^2}\exp(-y(i\omega Pr/\nu)^{\frac{1}{2}})\right]e^{i\omega t}. \tag{34}$$

Using the above solutions as the first approximations, further solution is possible, and the results of the second approximations are the same as those of the small amplitude case.

In (28) the convective terms are neglected to the first approximation under the conditions

$$\frac{\partial\theta_{pt}}{\partial t}\gg\frac{\partial\psi_t}{\partial y}\frac{\partial\bar{\theta}_p}{\partial x},\ \frac{\partial\bar{\psi}}{\partial y}\frac{\partial\theta_{pt}}{\partial x},\ \frac{\partial\psi_t}{\partial y}\frac{\partial\theta_{pt}}{\partial x} \quad \text{or} \quad \sigma\gg\epsilon\frac{\bar{\theta}_p}{\theta_{pt}},\ 1,\ \epsilon. \tag{35}$$

Moreover, an assumption is made that

$$\left(\frac{\partial^2\psi_t}{\partial y^2}\right)^2 \gg \frac{\partial^2\overline{\psi}}{\partial y^2}\frac{\partial^2\psi_t}{\partial y^2} \quad \text{or} \quad \sigma \gg \frac{1}{\epsilon^2}; \tag{36}$$

that is, ϵ is not too small. The simplified equation is

$$\kappa\frac{\partial^2\theta_{pt}}{\partial y^2}-\frac{\partial\theta_{pt}}{\partial t} = \frac{\nu}{c(T_w-T_\infty)}\left\{\left(\frac{\partial^2\psi_t}{\partial y^2}\right)^2-\overline{\left(\frac{\partial^2\psi_t}{\partial y^2}\right)^2}\right\} \tag{37}$$

and the solution is

$$\theta_{pt} = \frac{\epsilon^2 Pr\Gamma}{2(2-Pr)}[\exp\{-y(2i\omega Pr/\nu)^{\frac{1}{2}}\}-\exp\{-2y(i\omega/\nu)^{\frac{1}{2}}\}]e^{2i\omega t}. \tag{38}$$

It can be seen that the second harmonic fluctuation which is independent of x becomes predominant and the amplitude of fluctuating heat transfer is proportional to the square root of ω. Substituting (30), (38) into the left side of (27), we have

$$\frac{\partial\overline{\psi}}{\partial y}\frac{\partial\overline{\theta}_p}{\partial x}-\frac{\partial\overline{\psi}}{\partial x}\frac{\partial\overline{\theta}_p}{\partial y} = \kappa\frac{\partial^2\overline{\theta}_p}{\partial y^2}+\frac{\nu}{c(T_w-T_\infty)}\left\{\left(\frac{\partial^2\overline{\psi}}{\partial y^2}\right)^2+\overline{\left(\frac{\partial^2\psi_t}{\partial y^2}\right)^2}\right\}. \tag{39}$$

If we let

$$\overline{\theta}_p = \Gamma\{\overline{\theta}_{pa}+\overline{\theta}_{pb}\} \tag{40}$$

function $\overline{\theta}_{pa}$ can be reduced to $s(\eta)$ in (6) and $\overline{\theta}_{pb}$ to $W_h(x,y)$ in (17), and the time-independent result is the same as that of the small amplitude case in spite of the difference of the fluctuating results. Thus the assumption in (35) becomes clear as

$$\sigma \gg \epsilon\overline{\theta}_p/\theta_{pt} \quad \text{or} \quad \sigma \gg \epsilon^2.$$

5. Concluding remarks

The calculations described above are intended to demonstrate the large influence of viscous dissipation combined with high frequency oscillation on the time-mean heat transfer. The approximate formulae for large σ obtained for small amplitude oscillation, (21*b*) and (22*b*), are also valid for finite amplitude oscillation. In connexion with the high frequency results, the following brief discussions may be helpful. When the wall temperature is higher than the external flow temperature, the viscous dissipation effect may counteract the convection of heat from the wall. Then, from (20), the time-mean heat flux from the wall $\overline{q}_a$ is asymptotically given, for very high frequency and $Pr = 0{\cdot}72$, as

$$\frac{\overline{q}_a}{q_{0a}} = \frac{0{\cdot}4180-0{\cdot}3545\Gamma(1+2{\cdot}0315\epsilon^2(\sigma)^{\frac{1}{2}})}{0{\cdot}4180-0{\cdot}3545\Gamma} \quad (T_w > T_\infty) \tag{41}$$

in which q_{0a} is the heat flux without oscillation. It is anticipated that the reversal of heat flow can easily occur. The critical value of this reversal is obtained by letting $\overline{q}_a$ equal zero in (41) and is shown in figure 3, in which $\overline{q}_a > 0$ shows that the heat flows from the wall to gas and $\overline{q}_a < 0$ vice versa. When the external flow temperature is higher than the wall temperature, the heat generated by viscous dissipation may be superimposed on the convecting heat. Then, from the modification of (20), the heat flux to the wall $\overline{q}_b$ is given

$$\frac{\overline{q}_b}{q_{0b}} = \frac{0{\cdot}4180+0{\cdot}3545\Gamma_b(1+2{\cdot}0315\epsilon^2(\sigma)^{\frac{1}{2}})}{0{\cdot}4180+0{\cdot}3545\Gamma_b} \quad (T_\infty > T_w), \tag{42}$$

in which q_{0b} is again the heat flux without oscillation and $\Gamma_b = U_\infty^2/2c(T_\infty - T_w) = -\Gamma$. The ratio $\bar{q}_b/q_{0b}$ is shown in figure 4, and it can be seen that the heat flux for large Γ_b and $\epsilon^2(\sigma)^{\frac{1}{2}}$ can be several times as large as that without oscillation. These large influences of viscous dissipation can be explained mathematically as follows. The fluctuating friction is proportional to the square root of the frequency and it enters into the dissipation term of the energy equation in square form. Therefore it has a larger influence at higher frequency in the time-mean' energy equation.

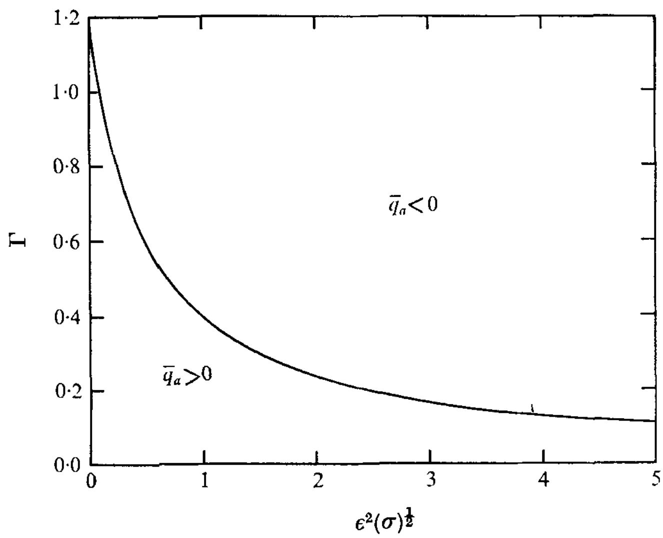

FIGURE 3. Critical value of heat flow reversal.

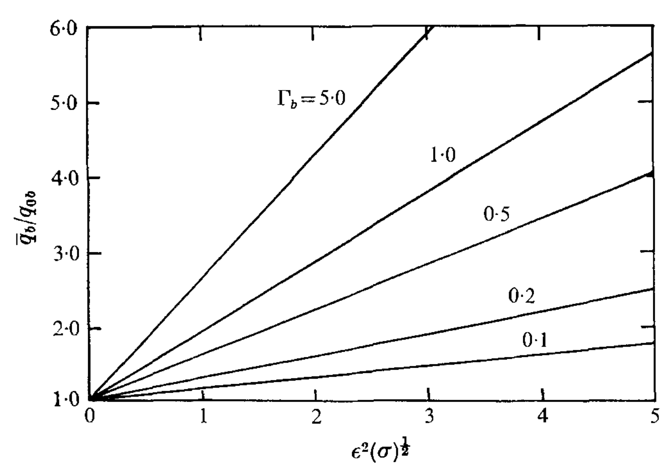

FIGURE 4. Increase of heat flux to the wall as a function of oscillation parameters for several values of Γ_b.

REFERENCES

FAGELA-ALABASTRO, E. B. & HELLUMS, J. D. 1969 *A.I.Ch.E.J.* **15**, 164.

GERSTEN, K. 1965 *AGARDograph*, **97**, 423.

ILLINGWORTH, C. R. 1958 *J. Fluid Mech.* **3**, 471.

ISHIGAKI, H. 1971 *J. Fluid Mech.* **46**, 165.

KEITH, H. G. & PURDY, K. R. 1967 *Proc.* 1967 *Heat Transfer & Fluid Mechanics Inst.* 298.

KESTIN, J., MAEDER, P. F. & WANG, W. E. 1961 *Appl. Sci. Res.* A **10**, 1.

LIGHTHILL, M. J. 1954 *Proc. Roy. Soc.* A **224**, 1.

LIN, C. C. 1957 *Proc. 9th Int. Congr. Appl. Mech.* **4**, 155.

MOORE, F. K. 1951 *NACA TN* 2471.

MOORE, F. K. & OSTRACH, S. 1957 *NACA TN* 3886.

OSTRACH, S. 1955 *NACA TN* 3569.

STUART, J. T. 1955 *Proc. Roy. Soc.* A **231**, 116.